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Contributions to the theory of elasticity by Louis Napoleon George Filon as viewed in the light of subsequent developments in biharmonic problems in applied mechanics and engineering mathematics

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Abstract. This paper contains introductory notes to the special issue of the Journal of Engineering Mathematics about the life and scientific work of L. N. G. Filon (1875–1937). The objective of this paper is to provide a complete description of the influence of Filon's two fundamental papers [1, 2] cited in the classical texbooks on the theory of elasticity by Love and Timoshenko and to document related contributions in various branches of engineering mathematics and mechanics conducted over the past twenty years. The biharmonic equation, which is a central aspect of Filon's works, has provided engineers and scientists with a wealth of avenues for the investigation of a variety of problems in applied mechanics and engineering mathematics dealing with the theory of bending of plates, two-dimensional and axisymmetric problems of stress analysis in solids and two-dimensional problems of slow viscous flow. Through the celebration of Filon's classical works we are also given an opportunity for examining the role of the biharmonic equation in the formulation and solution of problems in mechanics and applied mathematics.

Key words: biharmonic equation, elastic cylinder, elastic rectangle, L. N. G. Filon

1. Introduction

With this special issue of the *Journal of Engineering Mathematics* we celebrate the centenary of two fundamental contributions [1, 2] of L. N. G. Filon (1875–1937) to the development of the classical theory of elasticity. These papers contain mathematical statements of the boundary-value problems of the linear theory of elasticity that can be finally reduced to classical biharmonic problems for one scalar function of two coordinates with prescribed values of the function and its normal derivative at the boundary. In spite of the remark by Jeffery [3, p. 265], that all these biharmonic problems 'seem to be a branch of mathematical physics in which knowledge comes by the patient accumulation of special solutions rather than by the establishment of great general propositions', such problems were and still present challenging opportunities for applications in a variety of areas including the linear theory of elasticity, low-Reynolds-number hydrodynamics, structural mechanics of plates, and applied mechanics and mathematics.

We believe this anniversary offers us the opportunity to examine developments and applications of biharmonic problems in mechanics in the twentieth century.

2. Filon's benchmark memoirs

By the year 1901, Louis Napoleon George Filon, a rather shy young man full of strange enthusiasms (according to the recollections of his contemporaries), was the author of five papers (two in collaboration with the famous Karl Pearson) on a range of problems in mathematics and mechanics dealing with statistics, optics, elasticity and observational astronomy published in leading British journals of that time. In the late nineteenth century, the topic of torsion of elastic bodies had been examined by a number of eminent mechanicians including Coulomb, Navier and Saint-Venant. The torsion of non-circular prismatic bars in particular was examined by Saint-Venant in an inspired study [4] that led to the development of the inverse method [5–12]. Its publication represented both extensions and improvements to the theories proposed by Coulomb and Navier. Filon's most important work during the period that followed was a paper [13] on the distribution of shearing stresses on the torsion of elastic shafts, which was later to be cited by Love [7, Section 219], Timoshenko [14, Section 48], [15, Sections 75 and 79], and Timoshenko and Goodier [8, Section 106]. Filon had already discovered the direction in which his chief interest was to lie and in which he was to do his best studies. He was well qualified for work in this field and he knew its literature thoroughly. In a one year time interval, from May 20, 1901 to June 12, 1902 he submitted to the Royal Society two extensive papers on this topic. These papers (see also detailed abstracts [16, 17], which according to the traditions of the Royal Society at that time, were published very shortly after submission of the extensive manuscripts) are now regarded as benchmark studies in both engineering and applied mathematics. They dealt with practical problems of evaluation of stresses in short cylinders or rectangular plates either in tension, by applying a system of shearing forces along their cylindrical surface, or in compression between two absolutely rigid plates. Filon constructed approximate but reliable analytical solutions to these problems and discussed in length specific numerical results. (He constantly insisted that no mathematical results in physics, however elegant, were of any value unless it was carried through to the numerical evaluation of measurable quantities.) These two papers have ultimately been referred to in many textbooks on the theory of elasticity, and in recent research articles. A concise documentary of their contents may be of interest to the readers.

2.1. Filon's paper of 1902 on equilibrium of finite elastic cylinders and its subsequent developments

In this paper Filon addressed several problems concerning distributions of stresses and displacements in a circular elastic finite cylinder under certain axisymmetric systems of surface loadings which do not lead to the simple distributions of stress, usually assumed in practice. Filon [16, p. 354] stated:

The three problems investigated are as follows:

- In the first, I consider a cylinder under pull not being applied by a uniform distribution of tension across the plane ends, but by a given distribution of axial shear over two zones or rings, towards the ends of the cylinder.
- The second is that of a short cylinder compressed longitudinally between two rough rigid planes, in such a manner that the ends are not allowed to expand.
- The third case is that of the torsion of a bar in which the stress is applied, not by cross-radial shears over the flat ends, as the ordinary theory of torsion assumes, but by transverse shears over two zones or rings of the curved surface.

The analytical method employed solves the equations of elasticity in cylindrical coordinates, obtaining variables-separable solutions in the typical form $\begin{matrix}\cos\\ \sin\end{matrix}\{kz\} \times$ (function of r), with r being the distance from the axis and z the distance measured along the axis.

The *first* problem corresponds to conditions which frequently occur in tensile tests, namely, when the specimen is gripped by means of projecting collars, the pull in this case being transmitted from the collar to the body of the cylinder by a system of shear stresses. Filon constructed an approximate solution for a case where there is no radial pressure applied externally, and a uniform shear loading is applied between two zones. Although the solution gives zero resultant tension across the plane ends, it is found this arrangement cannot completely satisfy the condition of zero shear stresses on these planes. It is observed that a self-equilibrating system of shear stresses acts over the plane ends. The effect of these shear stresses diminishes with distance from the plane ends, which endorses the classical principle of Saint-Venant, assuming isotropy and homogeneity of the elastic material. The length of the cylinder is taken to be $\pi/2$ times the diameter. The two bands of shear loading each extend over one-sixth of the length and are at equal distances from the mid-section and the two ends. Filon [16, p. 355] proceeded:

> It is then found that the stress is greatest at the points where the shear is discontinuous, *i.e.*, at the ends of the collar in a practical case. At these points it is theoretically infinite. This result is true whatever the dimensions of the cylinder. For materials like cast iron or hard steel, which are brittle, such points would therefore be those of greatest danger; but in such a case as that of wrought iron or mild steel, for instance, the stress will be relieved by plastic flow.
>
> The tensile stress varies considerably over the cross-section, and the distortion of the latter is large. Towards the middle of the bar, the axial displacement at the surface is, roughly, twice what it is at the centre.
>
> In tensile experiments the elongation is usually measured by the relative displacement of two points on the outer skin of the cylinder, as recorded by an extensometer. When the test-piece is seized in this way, the surface stretches more than the interior, and consequently a negative correction should be applied to the readings of the extensometer. In the somewhat extreme case considered, this correction may amount to as much as 30 per cent.

Filon also presented a limited number of tables and figures to illustrate the values of the radial and axial displacements and of the four stresses for points in the cylinder at distances from the axis equal to 0, $0{\cdot}2a$, $0{\cdot}4a$, $0{\cdot}6a$, a; a being the radius of the cylinder; and for intervals of length parallel to the axis equal to tenths of the half-length; these results were briefly quoted in [18, Section 7.04]. The table for the axial stress in the finite cylinder, normalized with respect to the applied shear stress, has often been reproduced; for example, see Timoshenko's well-known textbooks, [14, Section 58], [15, Section 110], [8, Section 143], Lur'e's treatises [19, Section 6.3]; [20, Section 7.7], and a less known book [21, Chapter 6].

The motivation for the *second* problem is derived from the observation of the crushing of cylindrical blocks of cement or stone, when they are compressed between metal plates, such that their ends are constrained from movement; see also [7, Section 189] for a short description of the problem. The analytical solution is made up, partly in terms of a finite number of solutions, which are algebraic and rational in r and z, and partly of infinite series involving sines and cosines containing z. By suitably combining these two types of terms, the main boundary conditions can be satisfied. Tables of the stresses are given for a large number of points in the cylinder in which the length is nearly equal to the diameter (the exact ratio used was $\pi/3$, in order to minimize the numerical calculations in the pre-computer era). From these the principal stresses and the principal strains were calculated. Also, using interpolation, contours of the maximum stress, the maximum stretch, or the greatest principal stress-difference in the cylinder were calculated.

These curves show that, irrespective of the theory of yielding adopted, namely, the greatest-stress theory, or the greatest-strain theory, or the greatest-stress-difference theory, failure of elasticity will begin to take place around the perimeter of the plane ends. The fact that yielding first occurs at the perimeter, when the stress exceeds 1/1·686 of the limiting stress for uniform pressure, leads to the conclusion that the strength of a cylinder under this system of stress is considerably lower than the strength of a cylinder subject to uniform compression by normal traction alone.

The values of the apparent Young's modulus and of the apparent Poisson's ratio are investigated. Young's modulus is shown to vary between its true value, when the cylinder is long, and the value of the ratio of the axial stress to axial contraction, when lateral expansion is prevented by a suitable pressure, this last estimate corresponding to the case when the cylinder is made very short. The main results of the solution to this problem were later reproduced in [18, Section 7.08].

Finally, the *third* problem corresponds to the case of a cylinder whose ends are surrounded by a collar so that the applied torsional couple is transmitted to the inner core by means of shear stresses. As a numerical example, a cylinder whose length is $\pi/2$ times its diameter, is considered. A uniform transverse shear is applied over bands of its cylindrical surface, the width of these bands being one quarter of its length. Using the exact expressions obtained, the stresses and transverse displacement are calculated for various points, and these are compared with the values calculated from the approximate expressions when the cylinder is long. It is found that the agreement is reasonably good, whereupon it is inferred that, in torsion, the effect of local action dies out more rapidly than in cases involving either tension or compression.

Filon noted that in all his problems, when the applied transverse shear varies discontinuously, as in this case, the other stress becomes infinite at the points of discontinuity. This suggests the detrimental influences of abrupt changes in the section of the cylinder can contribute to failure. The projecting parts acting upon the inner core will introduce a sharp change in the applied shear stresses. It has been noticed that propeller shafts usually break at such points. This problem was cited by Love [7, Section 226B].

As a comment, Filon's paper [1] still remains citable: according to ISI database (January 2003) it has been cited 32 times since 1982, which is a creditable accomplishment for an article that appeared a hundred years ago! The citations mainly concerned testing experiments on various solid materials in structural [22–34], polymer [35–37], composite [38-45], rock [46–47], and biomechanics [48–53]. (It does not mean, however, that in earlier studies Filon's paper [1] went unnoticed; for example, the authors are aware of publication [54], and, probably, many more exist.)

Studies of an axisymmetrical equilibrium of an isotropic finite elastic cylinder published over the past century are too numerous to be mentioned individually. Among analytical approaches to the rigorous solution of the boundary-value problem we mention the method of eigenfunction expansion and the method of superposition. The first method represents a natural generalization of the classical expansion in scalar eigenfunctions for the Laplacian boundary-value problem to the vector boundary-value problems for the Lamé vector equation for displacement vector $\mathbf{u}(r, z)$. It consists in usage of a representation of this vector as a sum with yet undetermined complex coefficients C_s on the complex vector eigenfunctions $\mathbf{u}_s(r, z) = \mathbf{U}_s(r)\exp(\mathrm{i}\beta_s z)$ that leave the curved boundary $r = a$, traction-free. These 'transitionary modes' or 'homogeneous solutions', using a terminology of Dougall [55] or Lur'e [56], respectively, lead to a transcendental equation for β_s

$$(\beta a)^2 \left[I_1^2(\beta a) - I_0^2(\beta a)\right] + (2 - 2\sigma)I_1^2(\beta a) = 0, \tag{1}$$

where σ is Poisson's ratio. This equation was first obtained by Schiff [57]; see [58] for comments. Later Steklov [59] obtained a more complicated equation while considering an analogous problem for a hollow cylinder. Although the roots of the equation are not analyzed in any detail, it is mentioned that this equation has a trivial double zero root. (Remarkably, Steklov erroneously claimed that all other roots should be real and positive. Dougall [55, p. 939] correctly established that Equation (1) has an infinite number of complex roots and provided an approximate expression for large β_s, but he too was mistaken in stating that the equation has an infinite number of real roots.) In an additional note dated 3 October 1901, Filon [1, p. 151] correctly pointed out that the eigenfunctions-expansion approach to solve the problem suggested by Schiff [57] leads to a certain transcendental equation and non-orthogonal systems of functions that essentially complicate the solution from a numerical point of view. (It is worth noting that the expression on the left-hand side of Equation (1) appears in the denominator of some of Filon's analytical representations.) Vorovich [61] and Prokopov [60] gave a detailed overview of the twentieth-century developments in applications of the eigenfunctions method to various problems of equilibrium of elastic cylinders; see also [19, Chapter 7] for a typical example which also illustrates the amount of calculation to be done. Solution schemes which employ the eigenfunction-expansion methods were also considered by Little and Childs [62] and Flügge and Kelkar [63]. The participation factors in the eigenfunction-expansion schemes were obtained by employing a variety of numerical approximations, which do not always involve an infinite set of equations. Further methods have been formulated by Horvay and Mirabel [64] and Mendelson and Roberts [65].

The second analytical method employed in the solution of the problem for a finite cylinder is called the method of superposition. Lamé [66], in the twelfth of his famous lectures on the mathematical theory of elasticity, described this approach when considering the equilibrium of a three-dimensional elastic parallelepiped under any system of normal loads acting on its sides. Only briefly mentioned by Lamé [67, Section 102] as a possible method of solution of the two-dimensional scalar Laplace equation in a curved rectangle, this method for axisymmetric vector problems of an elastic equilibrium of a finite cylinder $0 \leq r \leq a$, $|z| \leq c$ was first addressed by Purser [68]. His paper went almost unnoticed, except for the study by Pickett [69], who considered, using this method, the second problem outlined in Filon's paper [1]. The main idea of the method consists in using the sum of two ordinary Fourier and Bessel-Dini series of the complete systems of trigonometric and Bessel functions in z and r coordinates, respectively, in order to represent an arbitrary displacement vector and stress tensor inside the cylinder. Each of these series satisfies identically the Lamé vector equation within the cylinder and has a sufficient functional arbitrariness for fulfilling the two boundary conditions, either on the curved boundary $r = a$ or the flat ends $z = \pm c$. Because of the interdependency, the expression for a coefficient of a term in one series will depend on all the coefficients of the other series and vice versa. Therefore, the final solution involves solving an infinite system of linear algebraic equations, thus providing finally the relations between the coefficients and loading forces. Interest in the method of superposition was revived only in the 1950's, when, almost simultaneously, the separate papers by Saito [70] and Abramian [71] were published. For the second problem of Filon [1], this method naturally gave poor results in the neighborhood of the circumference of the end planes. It is more than likely that several investigators have attempted to obtain solutions to this classical problem, perhaps either being unaware or, most likely, ignoring the thorny issue of stress singularities at the edges where the radial displacements are constrained. In an informative paper, Benthem and Minderhoud [72] examined the problem of the solid cylinder compressed between the rough rigid plane ends that constrain the radial displacements on these plane ends to zero, thereby correctly addressing the issue of the stress singularity at the bonded boundary. Due to the bonded conditions, the order of the stress singularity at the boundary of the adhered region is also influenced by the elasticity properties of the cylinder [73–78]. An exposition of the method of superposition and a detailed survey of studies based upon it (with further discussion of Filon's second problem, in particular) can be found in [79–83]. In this issue this approach is thoroughly addressed by several authors.

The aspect of decay of tractions in the elastic cylinder related to the third Filon problem was discussed further by Love [7] and von Mises [84] and a more formal proof of this variation on a theme by Saint-Venant was provided by Sternberg [85] (see also the review articles by Dzhanelidze [86], Gurtin [87, Sections 54–56*a*], and books by Fung [88], Lur'e [20, Section 2.8], and Davis and Selvadurai [89, pp. 180–187] for further references).

2.2. Filon's paper of 1903 on the equilibrium of an elastic rectangle and its subsequent development

In this paper Filon [2], addresses the elastic equilibrium of an isotropic elastic parallelepiped $x \leq |a|$, $y \leq |b|$, $z \leq |c|$, in those cases where the problem may be treated as two-dimensional one. A typical case is the one in which the loading is applied in the plane xy, with the thickness of the plate being in the direction of the z-axis. This thickness is small compared to the other dimensions of the plate, allowing us to approximate to the case of a thin plate under a thrust in its own plane. The starting line of this study was the theory of 'generalized plane stress' as it was later referred to by Love [7, Section 94]. The stresses in the plate, even under forces in its own plane, are not two-dimensional, for the stresses parallel to the plane xy vary through the thickness and the stresses normal to this plane usually vanish at the surfaces of the plate. Filon showed that if it is assumed that the normal traction across a face perpendicular to z is zero throughout the thickness (which will almost be true, the thinner the lamina), then the equations connecting the *mean* displacements U, V with the *mean* normal stresses P, Q and shear stress S in the plane of the plate (the mean here being taken in relation to the thickness of the plate) are of the same form as the Lamé equations in the traditional plane-strain case relating the actual displacement u, v with the three stresses in the plane of xy, provided only that we replace one of the Lamé elastic constants, by $\lambda' = 2\lambda\mu/(\lambda + 2\mu)$, where λ, μ are the elastic constants of Lamé. Of course, since the plate is thin, the displacements u, v will probably vary little as we move across it, so that the mean values U, V will give us an approximation to the displacements at every point. Similarly, the stresses in the planes parallel to xy will not differ greatly from their mean values P, Q, S. Thus Filon showed that the stresses *averaged through the thickness of the plate* could be calculated by an appropriate modification of the plane-strain two-dimensional equations, by merely changing the elastic constants, namely

$$\frac{\partial P}{\partial x} + \frac{\partial S}{\partial y} = 0\,, \qquad \frac{\partial S}{\partial x} + \frac{\partial Q}{\partial y} = 0\,, \tag{2}$$

where

$$P = \lambda' \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + 2\mu \frac{\partial U}{\partial x} ,$$
$$Q = \lambda' \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + 2\mu \frac{\partial V}{\partial y} , \qquad (3)$$
$$S = \mu \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) .$$

This opened the way for the solution of a wide class of problems, for two-dimensional problems are usually very much easier to solve mathematically than three-dimensional ones.

Performing interesting manipulations (see also [90, Section 48]) Filon found general solutions of Equations (2), (3) in terms of two arbitrary functions of the variable $\xi = x + \mathrm{i}y$ and two arbitrary functions of the variable $\eta = x - \mathrm{i}y$. Filon did not proceed, however, along this avenue which might have led him to the famous Kolosov-Muskhelishvili formulae (see [90, Section 49] for details) and the powerful method of complex variables in the two-dimensional theory of elasticity. Instead, in the first part of the memoir he established the formal solution for the general system of applied loading on the faces $y = \pm b$ with only the statical stress-resultants (total tension, total shear, total bending moment) at the faces $x = \pm a$ being given. These solutions were chosen in the form of Fourier series

$$\sum_{n=1}^{\infty} (a_n + b_n y) \left\{ \begin{matrix} \cosh \\ \sinh \end{matrix} \right\} \frac{n\pi y}{a} \times \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} \frac{n\pi x}{a} , \qquad (4)$$

with arbitrary constants a_n, b_n.

In a short historical discussion in Section 43 at the end of the paper, Filon mentioned that Ribière in his thesis [91] had obtained similar representations for stresses and displacements for a (long) elastic rectangle $0 \le x \le l$, $|y| \le b$ in the condition of plane stress. Ribière used a Fourier-series representation for the stress function on the complete system $\cos \frac{n\pi x}{l}$. In this way, initially, it appeared possible to satisfy exactly the boundary conditions over the sides $y = \pm b$. However, it was impossible to satisfy fully the conditions over the two short sides, $x = 0$ and $x = l$. Here $u = 0$, $v \neq 0$, $P \neq 0$, $S = 0$, and mechanically it corresponds to an infinite periodically-loaded strip with simple supports. If the ratio of the sides of the rectangle is large, it was believed (according to Saint-Venant's principle) that, at a large distance from the short ends, the effect of any self-equilibrated system of loads may be neglected, and the boundary conditions are fulfilled only for total tension, total shear and total bending moment. Independently, Belzeckii [92] used a similar approach, but with the complete system $\sin \frac{n\pi x}{l}$ in the Fourier series for the stress function. Here one has $u \neq 0$, $v = 0$, $P = 0$, $S \neq 0$ that corresponds to the conditions of 'free support'. Later Papkovich [93, pp. 412–464], presented a complete comparative analysis of the solutions of Ribière and Filon-Belzeckii for several cases of the loading of a rectangular plate. This served as a basis for a detailed study of some practical cases of bending of box-shaped rectangular empty beams which are widely used in shipbuilding.

Together with the infinite series (4), there enters into the solutions a finite number of terms of the form $c_{mn} x^m y^n$. These represent solutions for certain cases where Equations (2) can be solved in terms of polynomials, so as to give zero stress on the boundaries $y = \pm b$. For instance, a uniform tension parallel to the x-axis, a uniform bending moment, and a uniform shear give rise to solutions of this type. These can be superimposed without affecting the stress distribution over $y = \pm b$, and they are introduced to satisfy the terminal 'total' conditions.

When the length a of the beam is allowed to tend to infinity, the series degenerate into integrals. The transformation and interpretation of these integrals are dealt with at length. It is shown that they may be expanded in series of the form $\sum_n (d_n + e_n y) r^n \cos n\phi$, where r, ϕ refer to polar co-ordinates with the origin at any point in the beam, n is an integer, and d_n, e_n are constants, which are to be determined.

When the origin is a point on the surface $y = b$ where a concentrated load is applied, the series for the stresses start with a negative value of $n = -1$, giving terms which become infinite when $r = 0$. In this case the corresponding series for the displacements contain terms in $\log r$ and ϕ, which lead to discontinuities and singularities. These, of course, could not occur in any actual problem, since, in practice, the material in the vicinity of the concentrated load would probably become plastic, so that in the immediate neighbourhood of such loads, the solution will not apply. It is found that the terms involving infinities and discontinuities are precisely those of the well-known solutions given by Flamant [94] for two-dimensional problems for an elastic half-plane under a concentrated line force. The series of terms involving positive powers of r therefore represents the correction to Flamant's expressions, when the finite height of the beam is taken into account.

Filon considered several practical problems. The *first* problem concerns the case when the external loads at the top and bottom faces of the rectangle $y = \pm b$ are purely normal and are symmetrical about the mid-section, $x = 0$. The benchmark problem of a long beam lying on two supports and loaded by a concentrated force was considered in full. The results were compared with Saint-Venant's elementary theory and Stokes's empirical formula for the variation of stress in the mid-section. It is shown that, although the empirical formula gives an approximation to the stress in certain regions, it cannot be relied upon (see also [7, Section 245], [14, Sections 32, 42], [15, Section 32], [8, Section 40] for further discussion). The variations in the central deflection, as the supports are brought closer together, are also investigated. It is found that the discrepancy between the actual and the Euler-Bernoulli deflection (which excess is sometimes referred to by engineers as the 'deflection due to shear') decreases eventually as the span decreases and, for exceedingly small spans, may even become negative.

Filon studied the case of a beam under two opposite isolated loads, which leads to the more interesting problem of a beam $|x| \leq a$, $0 \leq y \leq b$ carrying an isolated load at the points $x = 0$, $y = b$ and resting upon a smooth rigid plane $y = 0$ capable of sustaining tensile tractions. The distribution of the pressure $Q(x, 0)$ upon the plane is investigated and a new form of expansion found for it. (This study was repeated in a slightly different manner in [18, Sections 5.07–5.10].) For a sufficiently long rectangle, this pressure becomes zero at $|x| = 1{\cdot}35b$ (almost independent on the ratio a/b for $a > 2b$), and the compressive pressure changes to tension. This result permits one to understand a simple experiment when an elastic block, acted upon by a concentrated load on its upper surface, cannot lie in full contact with a smooth rigid plane, unless the contact is capable of sustaining tensile tractions. At a certain distance from the force the ends separate from the plane. An accurate analysis of this remarkable phenomenon has to rely upon the solution of a complicated mixed-boundary-value problem where the boundary of the separation zone is in itself an unknown. An approximate estimate of the dimensions of the area in contact can be obtained by considering the area where the normal stresses Q are positive; see [78, 95, 96] for further discussion.

By combining a solution in the form of a Fourier series for a half-plane, Bleich [97] considered an interesting case when normal concentrated forces are applied at the centers of the short sides of a rectangle with $a < b$; see [15, Section 20] and [8, Section 24] for further details. This solution can be used for a quantitative estimate of the Saint-Venant principle: even for this extreme case the distribution of the stress Q over the cross-section is almost uniform for the distance a from the short ends.

The *second* problem is the case of a finite rectangular beam when the loads at $y = \pm b$ are still normal, but are asymmetrical with regard to $x = 0$. At the ends $x = \pm a$, shear stresses had to be applied to guarantee a static equilibrium. In particular, the behaviour of a beam under two concentrated loads acting in opposite senses on opposite faces of the beam, their lines of action being on opposite sides of the mid-section, is studied. The manner in which the shear stress across the middle section varies, as these loads are made to approach each other, is shown in several figures. They illustrate how rapidly the effects of the particular distribution of any total terminal load decay as we consider locations for from the end. At a distance of the same order as the height of the beam, they already begin to be negligible. These remarkable results were repeated in [18, Sections 5.07–5.10] and reproduced by Timoshenko [14, Section 42], [15, Section 20], [8, Section 24].

The *third* problem deals with the case when the loads at $y = \pm b$ are purely tangential, in particular, they are single concentrated tangential forces applied either at point $x = 0$, $y = b$ (with additionally applied normal and tangential forces at the ends $x = \pm a$, to ensure equilibrium) or symmetrically placed at four points $x = \pm c$, $y = \pm b$. For the latter case the correction to the readings of an extensometer (which measures the surface stretch), owing to the difference of this distribution of terminal stress from the one usually assumed, is investigated. It is found that errors will not be introduced, provided measurements are taken beyond a distance $2b$ from the grips. Remarkably, these results were reproduced only in the first textbook by Timoshenko [14, Section 42].

Finally, the *fourth* problem deals with the possible cases of solutions in the form of finite polynomials; such a solution is obtained for a beam which carries a uniform load. (This solution is usually attributed to Timpe [98], see also [14, Section 34], [15, Section 18], [8, Section 22].) It is shown that the assumptions of the usual theory of flexure are in this case no longer true, but are approximately true only if the height is very small compared with the span. The correction to the curvature as calculated from the usual formula is found to be a constant.

Filon's remarkable memoir [2] remains widely cited in the recent research literature: according to the ISI database (January 2003) since 1981 it has been cited 33 times in connection with the general theory of stress distribution in infinite elastic strips [99–104], more accurate evaluation of stresses in a finite rectangle [105–111], studies of cracks and fracture [112–118], testing of materials [119–123], anisotropic elasticity [124–128], and even in dynamics [129, 130].

3. Filon: his life and work

A complete commentary of the life and work of Louis Napoleon George Filon was given in the Obituary Note for the Royal Society records written by his student and later colleague G.B. Jeffery (also published in [131]). The ensuing excerpts contain some data from this commentary, in order to provide a more factual record of Filon's life and works.

3.1. Filon's short biography

Louis Napoleon George Filon (1875–1937), M.A., D.Sc., F.R.S., was the only son of Augustin Filon, the French littérateur who was tutor to the Prince Imperial. When Filon was three years old his parents (at this time his father was blind and his mother was in bad health) came to England. He began reading Latin and Greek before the age of six. Filon's ambition was to be a sailor. He was always drawing pictures of boats at sea and some of the models of ships he made at that time are still in existence. In later life this old ambition showed itself in his keen interest in the theory of navigation and in his one form of relaxation, namely, yachting. Filon graduated from University College, London, in 1896, obtaining his B.A. degree and receiving the Gold Medal for Greek. He was student of Karl Pearson and Micaiah J. M. Hill, two teachers for whom he had both affection and reverence. In 1898 Filon was elected to an 1851 Studentship and went to King's College, Cambridge. Here he published his benchmark studies on the theory of elasticity in which he developed the theory of 'generalized plane stress'. In May 1910, he was elected Fellow of the Royal Society. At the time of his election he was Assistant Professor of pure mathematics at University College, London. His proposers for the election included a veritable collection of pure and applied mathematicians and physicists of that time: M. J. M. Hill, K. Pearson, F. T. Trouton, A. E. H. Love, C. Chree, E. T. Whittaker, E. W. Hobson, G. F. C. Searle, J. J. Thompson and H. H. Turner. He went on to become Vice President of the Society. In 1912 Filon was appointed, as successor to Karl Pearson, to the Goldsmid Chair of Applied Mathematics and Mechanics at University College, London. Filon was on active service in France in the early months of World War I, but was recalled to command the 2nd (Reserve) Battalion London Regiment. Subsequently, he was appointed to the technical staff of the Admiralty Air Service. After World War I, Filon served as Vice-Chancellor of the University of London; his work was marked by academic freedom and the extension and development of teaching and research. The high offices in which he served and the heavy responsibilities that were laid upon him never led him to neglect his primary duty as a teacher. For years he carried a full lecturing time-table every morning and a full programme of committees and councils every afternoon. He was a man of strong convictions and his strongest conviction was the value of freedom. It was when he suspected an attempt to fetter the freedom of the university teacher and to make him a cog of a wheel in an administrative machine, that he fought with all his energy, neither asking nor giving quarter. Filon was elected a member of the London Mathematical Society in 1904, he was a Vice-President of the London Mathematical Society for the two years 1923–1925. He was also a Director of the University of London Observatory and Fellow of University College, London. Filon fell victim to the typhoid epidemic in Croydon, London and he died on 29 December, 1937.

Filon's expertise was in applied mathematics, classical mechanics and particularly the mechanics of continuous media. In assessing Filon's works in applied mathematics, it must be said that he was unsympathetic towards the 'modern developments' of this subject at that time. Although, the first edition of Whittaker's *A Course of Modern Analysis* (1902) had a great influence upon him. Filon loved to have a group of students working solidly through every example in the book. In this way, he attained mastery in the art of manipulating Legendre and Bessel functions – the ideal equipment for research in classical applied mathematics at that stage. (The later editions became too 'pure' for his liking.) His integration formula for highly oscillating trigonometric functions [132] is still in use [133, p. 890]. Filon wanted to rewrite classical mechanics as a collection of axioms, postulates, and propositions along the lines of Euclid [134, 135]. At the same time he wanted to treat it as a branch of experimental physics.

A large part of his time and energy were given to reconciling these two views. He came to the position that the fundamental principles must be established by quantitative experiment and that the subject must be erected as a logical structure on the foundations thus laid and again tested in every possible way by renewed appeal to experiment. Filon's lectures on mechanics were freely illustrated by experiment and he established a mechanics laboratory in which his students carried out experiments for themselves.

Relativity, particularly the 'general theory', seemed to him to rest on too slender an experimental basis. He regarded it as a kind of pure mathematics that had drifted out of touch with reality. For the quantum theorists he had nothing but scorn; their constantly changing hypotheses seemed to him to be sheer madness and the negation of that logical structure he expected to find in mechanics. As it was gradually borne in on him that the theory was actually producing good results, he found it very difficult to understand how this could possibly be.

A more detailed biography of Filon with a complete list of his scientific works consisting of 54 articles and 3 books can be found in [131]. This list, however, does not contain an interesting review [136] which Filon wrote of the treatise on Elasticity by Southwell, stressing 'that it is a storehouse of precious information which no scientific engineer and no mathematical physicist can afford to neglect.'

3.2. FILON'S STUDIES IN THE THEORY OF ELASTICITY

In Filon's time, the study of the mechanics of continua, particularly the theory of elasticity, was an important part of physical science. Jeffery [131, p. 315] wrote:

> It is a field that had been well culled over in previous time and the problems that were obvious and easy of solution had all been solved. Important and significant problems remained unsolved, but they usually presented formidable technical difficulties. Filon was well equipped both by temperament and by training to wrestle with such problems. He had a sound judgement in the choice of his subjects for research and turned to those which either yielded results of practical importance or were sufficiently general to advance the theory. His published work is singularly free from that multiplication of particular cases and artificial problems which is a special temptation to workers in this field. He had the mathematical courage that will tackle any problem, a resourceful mind and a pretty skill in meeting technical difficulties, and the patient perseverance that could hang on until the solution was reached.

Filon did important studies in several branches of engineering mathematics (as we refer to this area nowadays), but one can discover a central line of development beginning in his works [1, 2] on elasticity and culminating in his famous treatise on photo-elasticity [18] co-authored with Coker. This treatise was and still remains not only the compendium of a powerful experimental method for the exploration of the stresses in structures used in engineering practice (the main advantage is that, although experiments must necessarily be carried out with transparent material, the stress distribution is in certain circumstances independent of the material and thus experiments on glass or celluloid can give information about the behaviour of structures fabricated of materials such as steel), but also an exhaustive source of results on analytical methods of solution of the two-dimensional biharmonic equation in rectangular, polar, and elliptical coordinates. It also contains a thorough theoretical discussion of problems related to multi-connected two-dimensional elastic domains with openings and cracks. This book generalizes many of previous studies by Filon in the theory of elasticity scattered over some British journals that are hardly available at the present time. Among them there are:

– a further development [137] (see also [18, Sections 5.06, 5.15]) of the method of photo-elasticity which is based on the experimental fact that an isotropic transparent body, when stressed, becomes doubly refracting, with its optical principal axes at any point coincident with the directions of the principal axes of stress at the point (cited by Love [7, Sections 57, 245A] and Timoshenko [14, Sections 45, 64]);

– a discussion [138] (see also [18, Section 6.08]) of the theory of dislocations in the case of two-dimensional systems (cited by Love [7, Section 156A] and Timoshenko [15, Sections 35, 64], [8, Sections 43, 96]);

– a special case [139] (see also [18, Section 4.37]) of isostatic stress equations which hold in the case of plane stress, with application in photo-elasticity (cited by Love [7, Section 59] and Timoshenko [15, Section 38);

– a complete solution [140] (see also [18, Sections 4.32, 4.33, 5.17, 5.18]) of the two-dimensional problem on the stresses produced in a circular ring which is subjected to forces in its plane (cited by Love [7, Section 187] and Timoshenko [15, Section 36], [8, Section 44]).

We want, however, to draw attention to Filon's study [141] which went almost unnoticed. In this paper he considered the general problem of expanding a given function $f(x)$ in a series of functions $\phi(\kappa_r, x)$, where κ_r is the (real or complex) root of a transcendental equation $\psi(\kappa) = 0$. Based upon Cauchy's theory of residues, Filon established a general theorem for expanding a polynomial into a series of functions of the form $\phi(\kappa_r\, x)$. Next, he addressed the possibility of applying the method to a series of functions $\phi(\kappa_r,\ x)$ where κ_r and x do not appear exclusively as a product $\kappa_r\, x$. Referring to Dougall [142, Section 40], and considering the 'flexural' solution of the biharmonic equation in a semi-infinite strip $|x| \leq b,\ y \geq 0$, with free of loading faces $x = \pm b$, Filon arrived at a system of two functional equations that express the expansions of the prescribed normal $f(x)$ and shear loadings at $y = 0$ on the *two* systems of complex eigenfunctions with a *single* set of complex coefficients C_r. He proceeded to express explicitly (and *uniquely*, as he believed) the coefficients C_r by means of only *one* equation, provided $f(x)$ was a polynomial. He gave an example of such an expansion,

$$x^3 = \frac{3}{5}xb^2 - \sum_r \frac{b}{\kappa_r^2}\left[\kappa_r\, x\, \frac{\cosh \kappa_r\, x}{\sinh \kappa_r\, b} + (2 - \kappa_r\, b)\, \frac{\sinh \kappa_r\, x}{\sinh \kappa_r\, b}\right], \tag{5}$$

where κ_r is a complex root of the equation

$$\sinh 2\kappa b - 2\kappa b = 0\,, \tag{6}$$

and the summation extends to $\mathfrak{Re}\kappa_r > 0$.

This paradoxical mathematical result of the necessity of only *one* boundary condition $f(x)$ for normal loading, leaving the shear end stresses *arbitrary*, probably appeared so unusual to Filon (and, apparently, to many others), that almost no further papers were published on the subject for a long time. The single exception was the paper by Andrade [143] who noticed that Filon [141] used only one equation; however, no explanation was provided. This paradox was elucidated by Gomilko and Meleshko [144]. Further details and references related to the method of eigenfunction expansions for the biharmonic problem in a rectangle (or a half-strip), including the important bi-orthogonality property of these eigenfunctions that was established independently by Papkovich [145, 146] and Smith [147] (see also [148]), can be found in Meleshko [149, pp. 69–71]. Note that primary mathematical studies [150–154] need cautious consideration.

As we remarked earlier, the underpinning of Filon's contributions to the theory of elasticity is the use of the biharmonic equation for the formulation of the problems. The biharmonic equation in itself is a departure from the traditional fare in the treatment of partial differential equations in mathematical physics and engineering that invariably focus in the exhaustive study of Laplace's and Poisson's equations, the diffusion equation and the wave equation. The biharmonic equations usually arise as a result of the mathematical modelling of more detailed physical phenomena encountered in science and engineering. The exact first usage of the biharmonic equation is not entirely clear, since every harmonic function also satisfies the biharmonic equation. One of the earlier applications of the biharmonic equation deals with the classical theory of plates developed, among others, by J. Bernoulli, Euler, Lagrange, S. Germain, Poisson, Navier, and Cauchy. Applications of the biharmonic equation to the mathematical modelling of thin plates continued with the contributions of Kirchhoff (who correctly proposed the boundary conditions for the free edge of a thin plate), M. Lévy, Maxwell, and Lamb. Informative accounts of the historical developments in the area of flexure of thin plates are given by Kelvin and Tait [155], Nádai [156], Love [7], Timoshenko [9], Westergaard [157], Timoshenko and Woinowsky-Krieger [158], Truesdell [12], Bucciarelli and Dworsky [159], and Selvadurai [10].

The application of the biharmonic equation to the solution of two-dimensional problems of plane stress and plane strain in the classical theory of elasticity commences with the classical paper by Airy [160], who was Astronomer Royal and Director of the Greenwich Observatory, London. (He occupied this position from 1835 until 1881!) The Airy stress function was proposed in the course of an analysis of the structural supports for telescopes (see Meleshko [149, pp. 37–40] for further comments). These studies were followed by the works of Boussinesq, Hertz, and Love (see [9], [161]) dealing with the axisymmetric problem in the classical theory of elasticity, where the governing equations can be reduced to a single biharmonic equation for a scalar-valued function. An example of such a biharmonic function is Love's scalar potential [7, Section 188]. The biharmonic equation is also encountered in recent developments dealing with elasticity problems for inhomogeneous media developed by Spencer and co-workers [162–165]. In this innovative approach, a procedure is developed for obtaining exact solutions for the equations of linear elasticity for materials that are isotropic with an elastic inhomogeneity dependent on a specified coordinate direction.

The biharmonic equation is also encountered in the solution of two-dimensional problems dealing with slow viscous flows of a Newtonian fluid. With axisymmetric problems, the slow-viscous-flow problem yields a fourth-order partial differential equation for the Stokes stream function, whereas in two dimensions the governing equation for the stream function is a biharmonic one. This similarity between the two-dimensional problem for isotropic elastic behaviour and the two-dimensional problem for slow viscous flows has been exploited in a number of articles dealing with analogies dating back to Rayleigh [166]. Rayleigh also cites the work on slow viscous flows by Helmholtz and Korteweg, and proceeds to mention (on page 356) that 'Under the above restrictions, as is well known, the motion may be expressed by means of Earnshaw's current function ψ, which satisfies $\nabla^4\psi = 0$, the same equation as governs the transverse displacement of an elastic plate, when in equilibrium.' Others, including Goodier [167], Hill [168], Prager [169], Adkins [170] and Richards [171] (see also Meleshko [149, p. 36] for additional references), have used this observation to develop solutions to various problems involving plane stress and plate-bending problems in the infinitesimal theory of elasticity. This analogy has also been adopted in the formulation and solution of plane-strain problems in both finite elasticity [172, 173] and in second-order elasticity theory, where the governing partial differential equation for a displacement function is of a biharmonic type [174].

4. About this issue

This special issue brings together a collection of papers that addresses the importance of Filon's contributions to solid mechanics and mathematics and those that examine the continuing importance of the biharmonic equation in engineering mathematics in general and in solid mechanics, in particular.

The paper by Bespalova and Kitaygorodskii presents a method for the solution of the biharmonic equation, which is derived from the seminal studies by Kantorovich. The mixed analytical-numerical scheme is applied to problems involving clamped plates and to Filon's problem of the symmetric compression of a beam with a rectangular cross-section, by an equilibrating distribution of loads of finite width.

The paper by Davis deals with the rotational effects of Stokes flow when pressure-driven extrusion takes place in an annular opening in a rotating wall. The paper employs Abel-transform techniques to reduce the problem to three coupled integral equations. Formal results are presented to both limiting cases involving spatial dimensions and to quantities of engineering interest.

The paper by Flavin examines the spatial-decay estimates for a generalized biharmonic equation for elastic materials with spatial inhomogeneity in the elastic constants. Elastic inhomogeneity can occur naturally in geologic media and may be introduced purposely in functionally graded materials. The decay estimates are derived for different classes of elastic inhomogeneity applicable to a state of plane strain in a rectangular region where three adjacent sides are maintained traction-free.

The paper by Gomilko deals with the Dirichlet problem for the biharmonic equation for a semi-infinite strip. The superposition technique is an adaptation of the work of Lamé to two-dimensional plane problems. The analysis presents both formal results and general theorems applicable to the Dirichlet problem.

The application of a discrete double-Fourier-series method for solving plate-bending problems involving plates of variable thickness is presented by Grigorenko and Rozhok. The methodology proposed reduces the two-dimensional boundary-value problem to a one-dimensional problem by a suitable expansion into Fourier series in one variable. The presentation of the general method is followed by certain limited applications to a simply supported plate of variable thickness that is subjected to a double sinusoidal load.

The paper by Grinchenko is a survey of the approaches that have been adopted for the solution of both harmonic and biharmonic problems, including a problem examined by Filon.

Computational approaches to the solution of the biharmonic operator equations are gaining in popularity as the computational approaches are themselves being fine-tuned. The paper by Katsikadelis and Yiotis deals with the boundary-element modelling of plates of variable thickness that are elastically supported by a nonlinear two-parameter elastic foundation. The paper presents the computational developments associated with the title problem where the fundamental solution employed corresponds to the Green's function for the biharmonic equation. This paper also presents several useful examples involving elastically supported plates of rectangular and elliptical planforms, and with either clamped or simply supported boundary conditions.

The paper by Karnaukhov and Senchenkov examines the thermomechanical behaviour of a viscoelastic finite cylinder under harmonic deformations. The paper restricts attention to isotropic viscoelastic materials characterized by a complex shear modulus and constant Poisson ratio, thereby reducing the problem to a restricted class of materials. This facilitates the formulation of the three-dimensional dynamic loading of a finite cylinder since the eigenfunctions are independent of the viscoelastic parameters. The paper presents numerical results for practical situations involving rubber mounts used in vibration isolation.

One of Filon's lesser known works [138] relates to the observation how the difference between two solutions to the same traction boundary-value problem with the same shear modulus but different Poisson ratio, is related to the edge-dislocation problem with zero boundary traction. The paper by Knops is an authoritative extension of Filon's construction for an elastic wedge. The paper is noteworthy in its complete exposition of the background and its application of the complex-variable method in an elegant and economical fashion. The paper should be of considerable interest to student and researcher alike.

The paper by Meleshko re-examines Filon's problem for an elastic cylinder in the light of the above mentioned superposition method. The paper provides a complete documentary of the background of the problem, as well as comparisons of current results with those presented by Filon a century ago.

The paper by Selvadurai examines the problem of the axisymmetric loading of an annular crack by a disk inclusion placed centrally in the plane of the crack. The paper uses the Love strain potential, which satisfies the biharmonic equation, to reduce the annular crack-disk-inclusion interaction problem to that of a pair of coupled Fredholm integral equations of the second-kind. The numerical solution of these integral equations is used to develop results for the axial stiffness of the disk inclusion and for the Mode II stress intensity factors at the boundaries of the annular crack.

Ulitko and Lyakh examine the problem of a circular rigid shaft of a finite radius that is embedded in bonded contact with an isotropic elastic plane with a radial slit and subjected to a concentrated couple, which results in a finite rotation. The analysis presented is a very complete treatment of this innocuous-sounding problem that also accounts for the correct oscillatory form of the stress singularity at the boundary between the rigid shaft and the elastic medium at the free surface of the radial slit. The authors also present numerical results for the stress state and for the torsional stiffness of the embedded shaft along with a comparison with results provided by Neuber [175].

Finally, the paper by Villaggio deals with an approximate one-dimensional model of transferring force to an elastic half-plane from a thin elastic rod firmly attached to it horizontally.

5. Concluding remark

History, to paraphrase Leibniz, is a useful thing, for its study not only gives to those of the past their just due but also provides those of the present with a guide for the orientation of their own endeavors. It is hoped that this special issue of the Journal of Engineering Mathematics will help to honour Filon's works written at the beginning of the twentieth century and through these works provide a new impetus for the twenty-first century. Should the present issue achieve this objective, it will have accomplished its primary goal.

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